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Connection between classical and quantum harmonic oscillators

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Abstract. For a time-dependent harmonic oscillator we show a connection between the integrals of classical equations of motion and the Lewis–Riesenfeld (LR) invariant, and express the LR invariant in terms of the integrals of classical equations of motion. As applications we find the LR invariant for the Guth–Pi Hamiltonian and the asymptotic form of the LR invariant for a general time-dependent harmonic oscillator.

In recent years there has been a wide application of time-dependent quantum harmonic oscillators to quantum optics. Lewis–Riesenfeld [1] found an interesting invariant, a conserved quantity, for a time-dependent quantum harmonic oscillator in terms of whose eigenstates the exact evolution operator is completely determined up to time-dependent phase factors. Since then there has been enormous research on it; sometimes it is referred to as the Ermakov invariant [2–3], the generalized invariant [4–10], and the Lie algebraic method [11–13]. The Lewis–Riesenfeld (LR) invariant plays an important role in the evolution of time-dependent quantum harmonic oscillators. However, the exact evolution operators for time-dependent quantum oscillators have been found only for limited cases.

In this article, based on the Lie algebra so(2, 1) for a time-dependent harmonic oscillator, we shall show a connection between the integrals of classical equations of motion and the LR invariant. As applications we shall find the LR invariant for the Guth–Pi Hamiltonian and the asymptotic form of the LR invariant for a general time-dependent harmonic oscillator.

We consider a time-dependent classical harmonic oscillator of the form

$$H(t)=h_1(t)\tfrac{1}{2}p^2+h_2(t)pq+h_3(t)\tfrac{1}{2}q^2. \tag{1}$$

The Hamilton equations can be expressed by the vector equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}p(t)\\ q(t)\end{pmatrix}=\begin{pmatrix}-h_2(t) & -h_3(t)\\ h_1(t) & h_2(t)\end{pmatrix}\begin{pmatrix}p(t)\\ q(t)\end{pmatrix} \tag{2}$$

whose integrals of motion are formally given by

$$\begin{pmatrix}p(t)\\ q(t)\end{pmatrix}=T\exp\left[\int_{t_0}^{t}\begin{pmatrix}-h_2(t) & -h_3(t)\\ h_1(t) & h_2(t)\end{pmatrix}\mathrm{d}t\right]\begin{pmatrix}p(t_0)\\ q(t_0)\end{pmatrix}$$

$$=\begin{pmatrix}P_0(t,t_0) & P_1(t,t_0)\\ Q_1(t,t_0) & Q_0(t,t_0)\end{pmatrix}\begin{pmatrix}p(t_0)\\ q(t_0)\end{pmatrix} \tag{3}$$

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where T denotes time-ordered integration, and $P_0(t_0, t_0)=Q_0(t_0, t_0)=1$, $P_1(t_0, t_0)=Q_1(t_0, t_0)=0$. With the choice of basis

$$X_1=\tfrac{1}{2}p^2 \qquad X_2=pq \qquad X_3=\tfrac{1}{2}q^2 \tag{4}$$

the harmonic oscillator has a Poisson–Lie structure

$$\{X_1, X_2\}=-2X_1 \qquad \{X_1, X_3\}=-X_2 \qquad \{X_2, X_3\}=-2X_3. \tag{5}$$

Let us turn to the quantum harmonic oscillator with the Weyl ordering

$$\hat{H}(t)=h_1(t)\tfrac{1}{2}\hat{p}^2+h_2(t)\tfrac{1}{2}(\hat{p}\hat{q}+\hat{q}\hat{p})+h_3(t)\tfrac{1}{2}\hat{q}^2 \tag{6}$$

where the carets denote operators. It is well known that the quantum harmonic oscillator has a Lie algebra so(2, 1) with the basis [9]

$$L_1=\tfrac{1}{2}\hat{p}^2 \qquad L_2=\tfrac{1}{2}(\hat{p}\hat{q}+\hat{q}\hat{p}) \qquad L_3=\tfrac{1}{2}\hat{q}^2 \tag{7}$$

such that

$$[L_1, L_2]=-2\mathrm{i}L_1 \qquad [L_1, L_3]=-\mathrm{i}L_2 \qquad [L_2, L_3]=-2\mathrm{i}L_3. \tag{8}$$

There is a one-to-one map between the Poisson–Lie algebra and the Lie algebra under the correspondence: $\{\ ,\ \}\to -\mathrm{i}[\ ,\ ]$. The quantum evolution operator is formally given by

$$U(t, t_0)=T\exp\left[-\mathrm{i}\int_{t_0}^{t}\sum_{k=1}^{3} h_k(t)L_k\,\mathrm{d}t\right]. \tag{9}$$

We search for the LR invariant of the form

$$\hat{I}(t)=\sum_{k=1}^{3} g_k(t)L_k \tag{10}$$

which should obey the equation for an invariant (in units $\hbar=1$)

$$\frac{\mathrm{d}\hat{I}(t)}{\mathrm{d}t}=\frac{\partial\hat{I}(t)}{\partial t}-\mathrm{i}[\hat{I}(t), \hat{H}(t)]=0. \tag{11}$$

Equation (11) can be written by a linear system of the first-order differential equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} g_1(t)\\ g_2(t)\\ g_3(t)\end{pmatrix}=\begin{pmatrix} 2h_2(t) & -2h_1(t) & 0\\ h_3(t) & 0 & -h_1(t)\\ 0 & 2h_3(t) & -2h_2(t)\end{pmatrix}\begin{pmatrix} g_1(t)\\ g_2(t)\\ g_3(t)\end{pmatrix} \tag{12}$$

and the formal solution is given by

$$\begin{pmatrix} g_1(t)\\ g_2(t)\\ g_3(t)\end{pmatrix}=T\exp\left[\int_{t_0}^{t}\begin{pmatrix} 2h_2(t) & -2h_1(t) & 0\\ h_3(t) & 0 & -h_1(t)\\ 0 & 2h_3(t) & -2h_2(t)\end{pmatrix}\mathrm{d}t\right]\begin{pmatrix} g_1(t_0)\\ g_2(t_0)\\ g_3(t_0)\end{pmatrix}. \tag{13}$$

By introducing the independent matrices

$$N_1=\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & -\frac{1}{2}\\ 0 & 0 & 0\end{pmatrix}\qquad N_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\qquad N_3=\begin{pmatrix}0 & 0 & 0\\ \frac{1}{2} & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \tag{14}$$

we can show the connection $2N_k \to -\mathrm{i}L_k$ $(k=1,2,3)$ between the LR invariant and the quantum evolution operator.

Now the classical equations of motion for the basis of equation (4) are a linear system of the first-order differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\end{pmatrix}=\begin{pmatrix}-2h_2(t) & -h_3(t) & 0\\ 2h_1(t) & 0 & -2h_3(t)\\ 0 & h_1(t) & 2h_2(t)\end{pmatrix}\begin{pmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\end{pmatrix}. \tag{15}$$

By introducing an unitary matrix,

$$C=\begin{pmatrix}0 & 0 & 2\\ 0 & -1 & 0\\ 2 & 0 & 0\end{pmatrix}\qquad C^{-1}=\begin{pmatrix}0 & 0 & \frac{1}{2}\\ 0 & -1 & 0\\ \frac{1}{2} & 0 & 0\end{pmatrix} \tag{16}$$

we may rewrite equation (15) as

$$\frac{\mathrm{d}}{\mathrm{d}t}C\begin{pmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\end{pmatrix}=\begin{pmatrix}2h_2(t) & -2h_1(t) & 0\\ h_3(t) & 0 & -h_1(t)\\ 0 & 2h_3(t) & -2h_2(t)\end{pmatrix}C\begin{pmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\end{pmatrix}. \tag{17}$$

The fact that

$$\begin{pmatrix}g_1(t)\\ g_2(t)\\ g_3(t)\end{pmatrix}$$

satisfies the same equation for

$$C\begin{pmatrix}X_1(t)\\ X_2(t)\\ X_3(t)\end{pmatrix}$$

and the uniqueness theorem for the solution to a linear system of the first-order differential equations shows explicitly a connection between the integrals of classical equations of motion and the LR invariant. Therefore we obtain

$$\begin{pmatrix}g_1(t)\\ g_2(t)\\ g_3(t)\end{pmatrix}=C\begin{pmatrix}P_0^2 & P_0P_1 & P_1^2\\ 2P_0Q_1 & P_0Q_0+P_1Q_1 & 2P_1Q_0\\ Q_1^2 & Q_1Q_0 & Q_0^2\end{pmatrix}C^{-1}\begin{pmatrix}g_1(t_0)\\ g_2(t_0)\\ g_3(t_0)\end{pmatrix} \tag{18}$$

in terms of the integrals of motion of equation (3) for the classical harmonic oscillator.

As an example, we apply the connection (18) to the Guth–Pi Hamiltonian [14]

$$H(t)=\left(\frac{b}{2\pi}\right)^3\frac{e^{-3\chi t}}{2}\frac{p^2}{2}+2\left(\frac{2\pi}{b}\right)^3 e^{3\chi t}\{e^{-2\chi t}(k^2+\lambda^2)-m^2\}\frac{q^2}{2} \tag{19}$$

describing the decomposed modes of a scalar field in the expanding de Sitter universe, where b is the size for box normalization, χ is the expansion rate, k is a Fourier mode, and λ comes from the ground energy of the scalar field. The classical equation of motion

$$\frac{d^2q(t)}{dt^2}+3\chi\frac{dq(t)}{dt}-(m^2-e^{-2\chi t}(k^2+\lambda^2))q(t)=0 \tag{20}$$

has the integrals of motion of equation (3)

$$P_0(t,t_0)=\frac{\pi}{2}e^{3\chi(t-t_0)/2}\left[\left(\frac{3}{2}N_\upsilon(z)+z\frac{d}{dz}N_\upsilon(z)\right)J_\upsilon(z_0)-\left(\frac{3}{2}J_\upsilon(z)+z\frac{d}{dz}J_\upsilon(z)\right)N_\upsilon(z_0)\right]$$

$$P_1(t,t_0)=\pi\chi\left(\frac{2\pi}{b}\right)^3 e^{3\chi(t+t_0)/2}\left[\left(\frac{3}{2}N_\upsilon(z)+z\frac{d}{dz}N_\upsilon(z)\right)\left(\frac{3}{2}J_\upsilon(z_0)+z_0\frac{d}{dz}J_\upsilon(z_0)\right)\right.$$

$$\left.-\left(\frac{3}{2}J_\upsilon(z)+z\frac{d}{dz}J_\upsilon(z)\right)\left(\frac{3}{2}N_\upsilon(z_0)+z_0\frac{d}{dz}N_\upsilon(z_0)\right)\right] \tag{21}$$

$$Q_0(t,t_0)=\frac{\pi}{2}e^{-3\chi(t-t_0)/2}\left[J_\upsilon(z)\left(\frac{3}{2}N_\upsilon(z_0)+z_0\frac{d}{dz}N_\upsilon(z_0)\right)-N_\upsilon(z)\left(\frac{3}{2}J_\upsilon(z_0)+z_0\frac{d}{dz}J_\upsilon(z_0)\right)\right]$$

$$Q_1(t,t_0)=\frac{\pi}{4\chi}\left(\frac{b}{2\pi}\right)^3 e^{-3\chi(t+t_0)}[J_\upsilon(z)N_\upsilon(z_0)-N_\upsilon(z)J_\upsilon(z_0)]$$

where J_υ and N_υ are the Bessel functions of the first and second kinds, respectively, and

$$z=\frac{(k^2+\lambda^2)^{1/2}}{\chi}e^{-\chi t} \qquad \upsilon=\left(\frac{9}{4}+\frac{m^2}{\chi^2}\right)^{1/2}. \tag{22}$$

The LR invariant for the Guth–Pi Hamiltonian can now be read out by equation (18).

As a second application, we consider a time-dependent harmonic oscillator with a variable frequency of the general form

$$H(t)=\frac{1}{m}\frac{p^2}{2}+m\omega^2(t)\frac{q^2}{2} \tag{23}$$

whose classical equation of motion is

$$\frac{d^2q(t)}{dt^2}+\omega^2(t)q(t)=0. \tag{24}$$

The solutions may not be found explicitly in all general cases. However, we have at least the integrals of motion of equation (3) in an asymptotic form,

$$P_0(t, t_0)=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sin\int_{t_0}^{t}(\omega(t)+\delta_2(t))\,\mathrm{d}t}{\omega(t)+\delta_2(t)}\right]$$

$$P_1(t, t_0)=-m(\omega(t)+\delta_1(t))\sin\int_{t_0}^{t}(\omega(t)+\delta_1(t))\,\mathrm{d}t$$

$$Q_0(t, t_0)=\cos\int_{t_0}^{t}(\omega(t)+\delta_1(t))\,\mathrm{d}t \tag{25}$$

$$Q_1(t, t_0)=\frac{\sin\int_{t_0}^{t}(\omega(t)+\delta_2(t))\,\mathrm{d}t}{m(\omega(t)+\delta_2(t))}$$

where δ_1 and δ_2 are determined by the following perturbation equations:

$$\begin{aligned}\delta_1(t)=\delta_1(t_0)-\exp\left(-2\int_{t_0}^{t}\mathrm{d}t'\omega(t')\cot\int_{t_0}^{t'}\mathrm{d}t''(\omega(t'')+\delta_1(t''))\right)\\ \times\int_{t_0}^{t}\mathrm{d}t\exp\left(2\int_{t_0}^{t}\mathrm{d}t'\omega(t')\cot\int_{t_0}^{t'}\mathrm{d}t''(\omega(t'')+\delta_1(t''))\right)\\ \times\left[\omega(t)+\delta_1^2(t)\cot\int_{t_0}^{t}\mathrm{d}t'(\omega(t')+\delta_1(t'))\right]\end{aligned} \tag{26}$$

$$\begin{aligned}\delta_2(t)=\delta_2(t_0)-\exp\left(-\int_{t_0}^{t}\mathrm{d}t'\omega(t')\tan\int_{t_0}^{t'}\mathrm{d}t''(\omega(t'')+\delta_2(t''))\right)\\ \times\int_{t_0}^{t}\mathrm{d}t\exp\left(\int_{t_0}^{t}\mathrm{d}t'\omega(t')\tan\int_{t_0}^{t'}\mathrm{d}t''(\omega(t'')+\delta_2(t''))\right)\\ \times\left[\frac{\mathrm{d}\omega(t)/\mathrm{d}t}{\omega(t)+\delta_2(t)}+(\omega(t)+\delta_2(t))\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{\omega(t)+\delta_1(t)}\right)+\frac{1}{2}\delta_2^2(t)\right].\end{aligned}$$

Again the LR invariant for the Hamiltonian of equation (23) can be read out by equation (18).

By directly substituting

$$g_1(t)=\frac{\rho^2(t)}{m^2}\qquad g_2(t)=-\frac{\rho(t)}{m}\frac{\mathrm{d}\rho(t)}{\mathrm{d}t}\qquad g_3(t)=\left(\frac{d\rho(t)}{\mathrm{d}t}\right)^2+\frac{1}{\rho^2(t)} \tag{27}$$

we are able to rederive a single nonlinear equation [1],

$$\frac{\mathrm{d}^2\rho(t)}{\mathrm{d}t^2}+\omega^2(t)\rho(t)=\frac{1}{\rho^3(t)}. \tag{28}$$

Thus, equation (12) for the LR invariant is a linear system of the first-order differential equations corresponding to a single nonlinear equation such as equation (28).

In summary, the LR invariant of equation (10) for a time-dependent oscillator of equation (1) based on the Lie algebra so(2, 1) is given by equation (18) explicitly in terms of the integrals of the classical equations of motion of equation (3). The most prominent feature of the LR invariant thus obtained is that we no longer need to solve the nonlinear differential equation (28) but use instead the integrals of classical equations of motion, and the method used in this paper can be readily applied to other quantum systems with Lie algebraic structures such as time-dependent arbitrary spin systems.

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References

[1] Lewis Jr H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[2] Hartley J G and Ray J R 1981 *Phys. Rev.* A **24** 2873
[3] Goedert J 1989 *Phys. Lett.* A **136** 391
[4] Günter N J and Leach P G L 1977 *J. Math. Phys.* **18** 572
[5] Gordon T J 1986 *J. Phys. A: Math. Gen.* **19** 183
[6] Castaños O, Frank A and Lopez-Peña 1990 *J. Phys. A: Math. Gen.* **23** 5141
[7] Polio G and Soliani G 1991 *Phys. Rev.* A **44** 2057
[8] Gao X C, Xu J B and Qian T Z 1991 *Phys. Rev.* A **44** 7016
[9] Hirayama M 1991 *Prog. Theor. Phys.* **86** 341
[10] Gao X C, Xu J B and Gao J 1992 *Prog. Theor. Phys.* **87** 861
[11] Prants S V 1986 *J. Phys. A: Math. Gen.* **19** 3457
[12] Wolf F and Korsch H J 1988 *Phys. Rev.* A **37** 1934
[13] Fernandez F M 1989 *Phys. Rev.* A **40** 41
[14] Guth A H and Pi S-Y 1985 *Phys. Rev.* D **32** 1899